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GEODESIC FOLIATIONS BY CIRCLES

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1. Introduction

Smooth foliations by circles of compact three-manifolds have been completely analysed by D. B. A. Epstein in the paper [2]. Essentially, he shows that all such foliations arise as a decomposition of the manifold by the orbits of a smooth circle action. The theorem of this paper shows that the same is true of an arbitary smooth manifold, compact or not, with a foliation by circles satisfying a certain (rather strong) regularity condition.

It is known that not all foliations by circles arise as the orbits of some action by S^1; indeed, the paper [2] presents a foliated noncompact three-manifold as a counter-example to such a proposition. However, it is an open question whether or not such examples exist in the case of a foliated *compact* manifold of dimension greater than three.

A C^r flow on a C^r manifold M is a C^r action $\mu: \boldsymbol{R} \times M \to M$ of the additive reals on M. A C^r flow without fixed points, each of whose orbits is compact, gives rise to a C^r foliation of the manifold by circles. Further, any C^r foliation by circles of a manifold M gives rise to a C^r flow on (a double cover of) M. The version of the theorem presented here is stated for flows; an equivalent version for circle foliations in terms of differential forms is readily obtainable (see § 2). The theorem is the following.

Theorem. *Let $\mu: \boldsymbol{R} \times M \to M$ be a C^r action $(3 \le r \le \infty)$ of the additive group of real numbers with every orbit a circle, and M a C^r manifold. Then there is a C^r action $\rho: S^1 \times M \to M$ with the same orbits as μ if and only if there exists some riemannian metric on M with respect to which the orbits of μ are embedded as totally geodesic submanifolds of M.*

Finding some such metric given a circle action on M is easy (see § 3); the proof of the converse requires a little more effort. The author wishes to thank David Epstein for his gentle encouragement and for his many helpful suggestions.

2. The invariant one-form

Suppose a riemannian metric exists on the manifold M as in the theorem. At each point $m \in M$ choose a unit vector T_m in the direction of the flow μ.

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Then the vector field T satisfies the relations $|T| = 1$ and $\nabla_T T = 0$ on M, where ∇ is the Levi-Civita connection of the metric. Without loss of generality we may assume that the vector field T generates the flow μ. That is, $(d/dt)\mu_t(m)|_{t=0} = T_m$ where $\mu_t(m) = \mu(t, m)$.

Lemma 2.1. *Let $X \in T_mM$ and suppose that X is orthogonal to T. Then the vector $\mu_{t*}X$ in the tangent space of M at $p = \mu_t(m)$ for $t \in \mathbf{R}$ is orthogonal to T_p. That is, $\langle \mu_{t*}X, T \rangle = 0$ for all $t \in \mathbf{R}$.*

The proof of the lemma appears at the end of the section.

Thus the flow μ maps orthogonal vectors into orthogonal vectors for all time. Define a one-form α on M by $\alpha_m(X) = \langle X, T \rangle_m$; then $\alpha(T) = 1$ and $L_T\alpha = 0$, where L_T denotes Lie derivative with respect to the vector field T. This follows from Lemma 2.1 and the expression $(L_T\alpha) = \lim ((\mu_t{}^*\alpha)_m - \alpha_m)/t$ as $t \to 0$. In fact, we have a converse: given a vector field Y on M and a one-form β with $\beta(Y) = 1$ and $L_Y\beta = 0$ let $Q_m = \{X \in T_mM : \beta(X) = 0\}$ and $P_m = \{X \in T_mM : X = cY, c \in R\}$. Then the tangent bundle of M splits: $TM = Q \oplus P$. Furthermore, a straightforward construction defines a riemannian metric on M such that Q_m is orthogonal to P_m at each $m \in M$. The reverse argument to the proof of Lemma 2.1 (see below) then shows that with respect to this metric the trajectories of Y are geodesics in M.

In the formula $L_T\alpha = C_T(d\alpha) + d(C_T\alpha)$ where d is the exterior derivative and C_T is contraction by T, we have $d(C_T\alpha) = 0$, since $C_T\alpha = \alpha(T) = 1$. Whence $C_T(d\alpha) = L_T\alpha = 0$. Conversely, given a one-form β and vector field Y with $C_Y(d\beta) = 0$ and $\beta(Y) > 0$ it is easy to verify that $L_{Y'}\beta = 0$ and $\beta(Y') = 1$ where $Y' = Y/\beta(Y)$. We can summarise the above two paragraphs in the following

Lemma 2.2. *Let T be a nonzero vector field on the manifold M. Then there exists a riemannian metric on M so that the trajectories of T are embedded as totally geodesic submanifolds if and only if there exists a one-form α on M with $C_T(d\alpha) = 0$ and $\alpha(T) > 0$.*

Such one-forms arise naturally in the study of contact manifolds as defined by Boothby and Wang [1]. In this case, the manifold M is assumed to have dimension $2n + 1$ with a globally defined one-form ω such that $\omega \wedge (d\omega)^n \neq 0$ on M ($(d\omega)^n = d\omega \wedge \cdots \wedge d\omega$). On the subspace $V_x = \{X \in T_xM : C_X(d\omega) = 0\}$ we have $\omega \neq 0$; further, V_x has dimension one and is complementary to the subspace of dimension $2n$ on which ω is zero. Let Z_x be that element of V_x for which $\omega(Z_x) = 1$. Then the vector field Z and one-form ω satisfy the conditions of Lemma 2.2. Thus with a suitable metric on M the trajectories of Z are geodesics.

Indeed, in their paper [1] Boothby and Wang proved a special case of our theorem. They consider the case where the manifold M is compact and the induced foliation of M by the trajectories of Z is *regular* in the sense of Palais [6]. That is, about each point x of M there is an open neighborhood U of x so that any nonempty intersection of a trajectory with U is a connected set.

In this situation, each trajectory is closed and hence compact; so each orbit is a circle. They deduce that there is an *effective* circle action on M with the same orbits as the $\boldsymbol{R}$-action generated by Z.

Proof of Lemma 2.1. Suppose $X_m \in T_mM$ is orthogonal to the flow. Let V_0 be a small open disc transverse to the flow through m with cl V_0 compact (cl = closure), and X_m tangent to V_0 at m. Furthermore, assume there are defined on V_0 coordinate functions $x^2, \cdots, x^n$ ($n = \dim M$) with $x^i(m) = 0$ and $(\partial/\partial x^n)_m = X_m$. Then there is an $\varepsilon > 0$ such that $V = \mu((-\varepsilon, \varepsilon) \times V_0)$ is the diffeomorphic image of the open set $(-\varepsilon, \varepsilon) \times V_0$ under μ. Moreover, on V we may define coordinate functions $y^1, \cdots, y^n$ as follows: for $p = \mu_t(q)$ ($q \in V_0$, $-\varepsilon < t < \varepsilon$), set $y^1(p) = t$ and $y^i(p) = x^i(q)$, $2 \leq i \leq n$. Then $(\partial/\partial y^1)_p = T_p$ and $(\partial/\partial y^i)_p = \mu_{t*}(\partial/\partial x^i)_q)$; in particular, if $p = \mu_t(m)$ then $(\partial/\partial y^n)_p = \mu_{t*}X_m$. Define the vector field X on V by $X = \partial/\partial y^n$.

Because our hypotheses imply (i) $\nabla_T T = 0$, (ii) $\langle T, T\rangle = 1$ and (iii) $0 = [T, X] = \nabla_T X - \nabla_X T$, we have $T\langle X, T\rangle = \langle\nabla_T X, T\rangle + \langle X, \nabla_T T\rangle = \langle \nabla_T X, T\rangle = \langle \nabla_X T, T\rangle = \frac{1}{2}X\langle T, T\rangle = 0$. That is, the inner product $\langle X, T\rangle$ is constant along the orbit of T through m. In particular, we have $\langle \mu_{t*}X_m, T\rangle = 0$ for $-\varepsilon < t < \varepsilon$. So μ translates vectors orthogonal to the orbits of the flow into orthogonal vectors. This completes the proof. In general, the flow μ need not be metric-preserving.

3. Necessity

Let M be a riemannian manifold with metric tensor g. A vector field X on M which generates a one-parameter group of isometries of M with respect to g is known as a Killing vector field with respect to g. Such a vector field satisfies the condition $L_X g = 0$, where $L_X g$ is the Lie derivative of the tensor field g with respect to X.

Lemma 3.1. *Let M be a riemannian manifold with metric tensor g and a nonzero Killing vector field X. Then there exists a metric g' on M, conformal to g, such that X remains a Killing vector field with respect to g' and, in addition, we have $|X|' = 1$. Furthermore, with respect to g' the trajectories of X are geodesics with parametrisation by are-length.*

Proof. Define the function $f: M \to R$ by $f = (g(X, X))^{-1} = |X|^{-2}$. We may define the conformal metric g' by the tensor $g' = fg$. Now $L_X f = (g(X, X))^{-2} \cdot L_X(g(X, X)) = (g(X, X))^{-2}(L_X g)(X, X) = 0$, thus $L_X(fg) = (L_X f)g + f(L_X g) = 0$ because $L_X g = 0$ by hypothesis. The flow generated by X is isometric; in particular, the flow preserves the subspace of vectors orthogonal to X with respect to g'. It follows from § 2 that the trajectories of X are geodesics with parametrisation by are-length, as $|X|' = 1$.

Returning to the theorem, suppose we have $\rho: S^1 \times M \to M$, a smooth action of the circle group S^1 without fixed points. Identifying $S^1 = \boldsymbol{R}/\boldsymbol{Z}$, we may suppose ρ defines a flow with derived vector field T. Choose any metric g'' on M and define another metric by

$$g = \int (\rho^* g'') ,$$

where the integral is taken with respect to the invariant Haar measure on S^1. Then g is invariant under the action φ; that is, T is a Killing vector field with respect to g. Lemma 3.1 can now be applied to T thus proving necessity in the theorem.

4. Sufficiency

Suppose that we are given a flow $\mu: \boldsymbol{R} \times M \to M$ with every orbit a circle, and that with respect to some riemannian metric on the manifold M the orbits of μ are geodesics. Without loss of generality we may suppose parametrisation by arc-length. By Lemma 2.1 we see that the flow maps orthogonal vectors into orthogonal vectors.

Let V_0 be a small disc in M transverse to the flow, with cl V_0 compact. Then there is an $\varepsilon > 0$ such that μ defines a homeomorphism of $[-\varepsilon, \varepsilon] \times \text{cl } V_0$ into M, which is a diffeomorphism on $(-\varepsilon, \varepsilon) \times V_0$. By a *flat neighborhood* in M (resp. of a point m in M) shall be meant an open subset V of M (resp. an open neighborhood V of m) such that $V = ((-\varepsilon, \varepsilon) \times V_0)$ for some disc V_0 (resp. for some disc V_0 with $m \in V_0$). Let $\pi: V \to V_0$ be the projection map.

Lemma 4.1. *Let V be a flat neighborhood in M. Let $\sigma_1: [0, 1] \to V$, $\sigma_2: [0, 1] \to V$ be smooth curves in V orthogonal to the flow. If $\pi \circ \sigma_1 = \pi \circ \sigma_2$ and $\sigma_1(0) = \sigma_2(0)$, then $\sigma_1 = \sigma_2$.*

Proof. A straightforward application of the uniqueness of solutions of ordinary differential equations.

Following [2, p. 69], we define $\lambda: M \to \boldsymbol{R}$ by the conditions

$$\begin{aligned} &\text{i.} \quad \lambda x > 0 , \\ &\text{ii.} \quad \mu_t(x) \neq x \qquad \text{for } 0 < t < \lambda x , \\ &\text{iii.} \quad \mu_{\lambda x}(x) = x . \end{aligned}$$

The function λ is invariant under the flow.

Proposition 4.2, [2, § 5]. *The function $\lambda: M \to \boldsymbol{R}$ giving the period of a point is lower semi-continuous. If $W \subset M$, then the set of points of continuity of $\lambda | W$ is open in the induced topology on W.*

We now use an idea basically due to Montgomery (see [4, p. 224]). We define the sets $B_1, B_2 \subset M$ as follows

$$B_1 = \{x \in M : \lambda \text{ is not continuous at } x\} ,$$
$$B_2 = \{x \in B_1 : \lambda | B_1 \text{ is not continuous at } x\} .$$

Each of B_1, B_2 is invariant. Furthermore, B_1 (resp. B_2) is closed and has null interior as a subspace of M (resp. B_1). $M - B_2$ has a countable number of connected components each of which is an invariant open subset of M.

Lemma 4.3. *Let U be an open connected set in M, and $f: U \to \boldsymbol{R}$ a continuous, invariant real-valued map such that $\mu_{fm}(m) = m$ for all $m \in U$. Then f is a constant map.*

Proof. Fix $x \in U$. Let $V = \mu((-\varepsilon, \varepsilon) \times V_0)$ be a flat neighborhood of x in M. Then on V, $\lambda \geq 2\varepsilon$. Choose another neighborhood W of x, $W = \mu((-\varepsilon, \varepsilon) \times W_0)$, $x \in W_0 \subset V_0$ such that for $y \in W$ we have $|fx - fy| < \varepsilon$. For $p' \in W$, by taking a smaller neighborhood if need be, we may further suppose that there exists an orthogonal curve $\sigma: [0, 1] \to W$ with $\sigma(0) = x$ and $\sigma(1) = p$, where p and p' lie on the same connected component of an orbit in W. Now $\mu_{fx} \circ \sigma$ is orthogonal and its image is contained in W; furthermore, it is easy to see that $\pi \circ \sigma = \pi \circ (\mu_{fx} \circ \sigma)$ where $\pi: W \to W_0$ is projection. Since $\sigma(0) = x = \mu_{fx} \circ \sigma(0)$ we may apply Lemma 4.1 to obtain $\sigma = \mu_{fx} \circ \sigma$. In particular, $\mu_{fx} \circ \sigma(1) = \mu_{fx}(p) = p$. Clearly $fp = k_1 \lambda p$ where k_1 is an integer; similarly, we have $fx = k_2 \lambda p$. As $|fx - fp| < \varepsilon$ and $\lambda p \geq 2\varepsilon$ we obtain $|k_1 - k_2| < \frac{1}{2}$, which implies $k_1 = k_2$. Whence $fx = fp = fp'$. As $p' \in W$ was arbitary and U is connected, the lemma is proved.

Corollary 4.4. *Let U be a connected component of $M - B_1$. Then $\lambda | U = c$, a constant.*

Define $C_1 = \{x \in M : \lambda$ is unbounded in any neighborhood of $x\}$. C_1 is a closed invariant subset of M. Furthermore, we have $C_1 \subset B_1$ as the function λ is locally constant on $M - B_1$. In the proof we assume C_1 is nonempty and prove a contradiction.

Proposition 4.5. *Let D be a connected component of $M - C_1$. If $U \subset D$ is a component of $M - B_1$ with $\lambda | U = c$, then $\mu_c | D = \mathrm{id}$.*

Proof. D is an open invariant subset of M. Fix $m \in D$ and let $A \subset D$ be the orbit of μ through m. Let $V = \mu((-\varepsilon, \varepsilon) \times V_0)$ be a flat neighborhood of m in D, so $\lambda \geq 2\varepsilon$ on V and cl V_0 is compact. Because λ is locally bounded on D, we may assume that $\lambda \leq \Lambda$ on V, $\Lambda \in \boldsymbol{R}$. Additionally, it can be supposed that the disc V_0 is sufficiently small to ensure that the orbit A intersects V_0 in only the single point m. We define the Poincaré map $S: V_1 \to V_0$ for some smaller disc $V_1 \subset V_0$. For more detail the reader is referred to [2, §§ 4, 5]. Essentially, there exists a neighborhood V_1 of m in V_0 such that the map $f: V_1 \to \boldsymbol{R}$, given by the conditions

$$\begin{aligned} &\text{i.} \quad fx > 0\,, \\ &\text{ii.} \quad \mu_t(x) \notin V_0 \qquad \text{for } 0 < t < fx\,, \\ &\text{iii.} \quad \mu_{fx} \in V_0\,, \end{aligned}$$

is well-defined and C^r on V_1. The Poincaré map $S: V_1 \to V_0$ is defined by $S_x = \mu_{fx}(x)$. The point $m \in V$ is invariant under S. Let $N = [\Lambda/(2\varepsilon) + 1]$.

We define by induction neighborhoods V_i of m in V_0 such that $SV_{i+1} \subset V_i$ $(1 \leq i \leq N!)$. Because $\lambda \geq 2\varepsilon$ on the open invariant set orb V_0 (where orb $V_0 = \{y \in M : y = \mu_t(x)$ for $t \in \boldsymbol{R}, x \in V_0\}$) and because $\lambda \leq \Lambda$ here, it is easy to show that for each point $x \in V_q$, where $q = N!$, $S^r x = x$ for some r, $1 \leq r \leq N$. Hence $S^q =$ id on V_q. We obtain an open neighborhood W of m in V_0 which is invariant under S by putting $W = \bigcap_{i=1}^q S^i V_q$. The set orb $W \subset D$ is invariant, connected and open in M. Define the function $g : W \to \boldsymbol{R}$ by

$$g(x) = \sum_{i=1}^{q} (f \circ S^i x) .$$

Then g is continuous and invariant under S. Thus it may be extended continuously to a function g on all of orb W, invariant under μ and agreeing on W. Because $S^q =$ id on W we have $\mu_{gx}(x) = x$ for every $x \in$ orb W. By Lemma 4.3, g must be constant on orb W. As the set $M - B_1$ is open and dense in M, some component U of $M - B_1$ intersects orb W nontrivially. Let $\lambda | U = c$. It is easy to see that $g = kc$ on W, where k is some integer, and thus $g = kc$ on orb W. The transformation $\mu_c |$orb W is periodic and is the identity on the interior set U orb W. By a theorem of Newman [5], $\mu_c |$orb $W =$ id. Straightforward use of a covering of D by flat neighborhoods and the fact that D is connected completes the proof of the proposition.

Corollary 4.6. *For D as above, $\mu_c |$cl $D =$ id, and if $x \in$ cl D then we have $k_x \lambda x = c$ where $k_x \geq 1$ is an integer. Furthermore, $\mu_{c*} : T_x M \to T_x M$ is the identity for each $x \in$ cl D.*

Corollary 4.7. *For D as above we have* bdy $D =$ bdy (cl D); *that is,* int (cl D) $= D$.

It will be useful to consider the action μ on the component D of $M - C_1$, where $\mu_c | D =$ id as above. Define another metric g'' on M by

$$g'' = c^{-1} \int_0^c (\mu_t{}^* g) dt .$$

It follows from Corollary 4.6 that on cl D the flow is isometric with respect to g''. It will be convenient to work with the g''-metric only for the remainder of the proof.

Since μ is isometric on the open set D, it commutes with the exponential map there. For $p \in M$, $r > 0$ set $B'_r(p) = \{X \in T_p M : |X|'' < r\}$ and define $B_r(p) = \exp B'_r(p)$. If $p \in D$, then there exists some $r > 0$ such that $B_r(p) \subset D$ and $B_r(p)$ is the diffeomorphic image of the ball $B'_r(p)$ in $T_p M$. Thus $\mu_t \circ \exp_p | B'_r(p) = \exp_q \circ \mu_{t*} | B'_r(p)$ for $q = \mu_t(p)$ and all time t; in particular, the set $B_r(p) = \mu_{\lambda p} B_r(p)$, so that the action of $\mu_{\lambda p}$ in a neighborhood of p is linear with respect to geodesic coordinates at p.

It follows from Proposition 4.2 that if $m \in (B_1 - B_2) \cap D$ then there exists a neighborhood W of m in D such that $\lambda | B_1 \cap W$ is continuous. By choosing some smaller neighborhood if necessary, we can suppose $\lambda | B_1 \cap W$ is constant. (Because $\mu_c | \mathrm{cl}\, D = \mathrm{id}$ and λ is locally bounded below, we may first suppose that $\lambda | B_1 \cap W$ takes only a finite set of values. Then, since λ is continuous on this set, we can easily find a (smaller) neighborhood W' of m so that $\lambda | B_1 \cap W'$ is constant.) Suppose $\lambda m = c/k$, $k \geq 1$ an integer. Then the transformation $\mu_{\lambda m *}: T_m M \to T_m M$ is such that every vector is either fixed or has period k. Using the diffeomorphism $B_r(m) = \exp B'_r(m)$ it is easy to see that if $k = 1$ then λ would be continuous at m, whence $k \geq 2$; thus the fixed point set of $T_m M$ (with respect to $\mu_{\lambda m *}$) has codimension at least one. Denote this set by $H'(m)$ and define $H(m) = \exp_m H'(m)$. Thus $\mu_{\lambda m} | H(m) \cap B_r(m) = \mathrm{id}$ and the only fixed points of $B_r(m)$ under the transformation $\mu_{\lambda m}$ are contained in $B_r(m) \cap H(m)$. (Note that $B_r(m) \cap B_1$ possibly includes points of B_2.)

Define $C_2 = \{x \in C_1 : \lambda | C_1 \textit{ is} \text{ continuous at } x\}$. By Proposition 4.2, C_2 is an open subset of C_1 (with respect to the relative topology). Let $p \in \mathrm{bdy}\, D \cap C_2$ where D is some component of $M - C_1$. ($\mathrm{bdy}\, D \subset C_1$ because points of $\mathrm{bdy}\, D$ are not interior in $M - C_1$.) Then there exists a neighborhood W of p in M such that $\lambda | W \cap \mathrm{bdy}\, D$ is continuous and, as before, we may suppose that λ is constant there.

Lemma 4.8. $\lambda | \mathrm{bdy}\, D \cap W = c$.

Proof. Most of the work in the proof of this lemma arises because $\mathrm{bdy}\, D$ need not *a priori* be a smoothly embedded submanifold of M.

Without loss of generality, the point p is arcwise accessible from D; that is, there is some (regular) arc lying in $D \cup \{p\}$ having p as an endpoint. Such points are obviously dense in the boundary (see, for example, [4, p. 119]). With a slight abuse of notation, denote some such arc by $[q, p]$ with $[q, p)$ contained in $W \cap D$.

It is well-known that given any compact set $A \subset M$ there exists an $s > 0$ such that for each $x \in A$ the ball $B_s(x)$ is convex and such that if the vector $X \in T_y M$, $y \in \mathrm{bdy}\, B_s(x)$ is tangent to the sphere $\mathrm{bdy}\, B_s(x)$ then the geodesic $\exp tX$ does not penetrate the ball $B_s(x)$ near y (see, for example, [3, § 9.4]). Setting $A = [q, p]$ we let $s > 0$ as above; we may further suppose that $B_s(q) \subset D$ and that if $x \in [q, p]$ then $\mathrm{cl}\, B_s(x) \subset W$. Then there exists some $y \in [q, p)$ such that $\mathrm{bdy}\, D \cap \mathrm{cl}\, B_s(y) \neq \emptyset$ and $\mathrm{bdy}\, D \cap \mathrm{cl}\, B_s(y) \subset \mathrm{bdy}\, B_s(y)$. Let $z \in \mathrm{bdy}\, D \cap \mathrm{bdy}\, B_s(y)$. If $X \in T_z M$ is tangent to the sphere $\mathrm{bdy}\, B_s(y)$ then the geodesic $\exp tX$ lies outside of $B_s(y)$ near z; furthermore, if X is not tangent to this sphere, then the geodesic $\exp tX$ or $\exp(-tX)$, $t > 0$ penetrates the ball $B_s(y)$ for some positive distance. Note that as the radius s varies over lesser values such points z will be arbitrary near p, and that $\lambda p = \lambda z$.

There are a metric ball $B_r(z) \subset W$ with center z and an $a > 0$ such that if $x \in B_r(z)$ then the ball $B_a(x)$ is convex. Thus the open set $B_a(z) \cap B_s(y)$ is convex and contained within $D \cap W$. We may distinguish two cases:

1. z is approximated by points of $B_1 - B_2$ in $B_a(z) \cap B_s(y)$,
2. case 1 does not occur.

Consider case 1. Let m_1 be an element of $B_1 - B_2$ in $B_a(z) \cap B_s(y)$ with associated fixed-point set $H(m_1)$ (see the paragraph following Corollary 4.7). Recall that the flow, when restricted to D, preserves the metric and consequently maps geodesics into geodesics whilst preserving their parametrisation. In particular, if $g\colon [0, 1] \to \operatorname{cl} D$ is a geodesic with $g(0) = x \in \operatorname{bdy} D$ and $g(0, 1] \subset D$, then for each integer k we have $\mu_{k\lambda x} \circ g\colon [0, 1] \to \operatorname{cl} D$ is a geodesic on $(0, 1]$ and, by continuity, it must be geodesic at $\mu_{k\lambda x} \circ g(0) = x$. Now, if the set $H(m_1) \cap B_a(m_1) \cap B_a(z)$ intersects bdy D then it contains an open subset of bdy D, which is impossible. For otherwise, there is some $w \in \operatorname{bdy} D \cap B_a(m_1)$, $w \notin H(m_1)$ with a geodesic $\exp tX$ in $B_a(m_1)$, X tangent to M at m_1, such that $\exp t_0X = w$ and $\exp tX \in D \cap B_a(m_1)$ for $0 < t < t_0$. Thus the point $\exp t_0X$ is fixed under the transformation $\mu_{\lambda m_1}$. By the definition of $H(m_1)$ we have $t_0X \in H'(m_1)$ which contradicts the hypothesis that $w \notin H(m_1)$.

Furthermore, $H(m_1) \cap \operatorname{cl} B_a(m_1) \cap B_a(z)$ is closed in $B_a(z)$, and is therefore bounded away from z. Thus we may choose $m_2 \in (B_1 - B_2) \cap (B_a(z) \cap B_s(y))$ strictly nearer z than m_1 so that $m_2 \notin H(m_1)$ but $m_2 \in B_a(m_1)$. (Because $B_a(m_1)$, $B_a(z)$ are convex and $z \in B_a(m_1)$.) Proceeding inductively, we may find $m_i \in (B_1 - B_2) \cap (B_a(z) \cap B_s(y))$ strictly nearer z than m_{i-1} with $m_i \notin H(m_j)$, $1 \leq j < i$ but with $m_i \in B_a(m_j)$, $1 \leq j < i$. By the definition of $H(m_j) \subset B_a(m_j)$ we have $\lambda m_i \neq \lambda m_j$ for $1 \leq j < i$. But, by hypothesis, λ is bounded away from zero in W and $B_a(z) \subset W$. Moreover, $\mu_c | \operatorname{cl} D = \mathrm{id}$. Hence there is only a finite number of values for $\lambda | D \cap B_a(z)$. In particular, $\lambda | B_1 \cap D \cap B_a(z)$ takes only finitely many values; but this contradicts the construction of our sequence $\{m_i\}$. Thus case 1 cannot occur.

Consider case 2. That is, z is not approximated by points of $B_1 - B_2$ in $B_a(z) \cap B_s(y)$. But since $B_1 - B_2$ is open and dense in B_1, for some smaller value of a we also have that $B_1 \cap (B_a(z) \cap B_s(y)) = \emptyset$. Thus $B_a(z) \cap B_s(y) \subset U$, where U is some component of $M - B_1$, $U \subset D$ and $\lambda | U = c$. By Corollary 4.6, if $x \in \operatorname{bdy} D \cap W$ then $\lambda x = kc$ where $k \geq 1$ is an integer. Consider the case $k \geq 2$.

In T_zM denote by F the one-codimensional hyperplane of vectors tangent to the sphere bdy $B_s(y)$. F partitions T_zM into two complementary open half-spaces E^- and E^+ where E^+ consists of vectors X such that the geodisic $\exp tX$, $t > 0$, penetrates the ball $B_s(y)$ for some positive distance. Restricting attention to the ball $B_a(z)$, for each vector $X \in E^+$ and integer j the curve $\mu_{j\lambda z} \circ \exp tX$ (for small $t > 0$) is a geodesic in D; consequently, $\mu_{j\lambda z} \circ \exp tX = \exp_z \circ\, t(\mu_{j\lambda z *}X)$. Since $\mu_c | U = \mathrm{id}$, each vector X in E^+ has period k with respect to $\mu_{\lambda z *}\colon T_zM \to T_zM$. If there is $t_0 > 0$ such that $\exp t_0X \in \operatorname{bdy} D \cap B_a(z)$ where $X \in E^+$, then the vector t_0X would be fixed under $\mu_{\lambda z *}$ (as $\lambda | W \cap \operatorname{bdy} D$ is constant) which contradicts the fact that X has period $k \geq 2$. Thus $\exp_z$ maps $E^+ \cap B'_a(z)$ diffeomorphically into $U \subset D$.

Lemma 4.9. *Let V be a vector space with $F \subset V$ a one-codimensional hyperplane, and $T\colon V \to V$ a linear transformation of finite period such that* *each point of the open half-spaces E^-, E^+ of V determined by F has least period (with respect to T) strictly greater than one. Then the orbit of E^+ under successive transformations by T includes E^-.*

Proof. It is sufficient to show that the orbit of each point $v \in E^-$ intersects E^+ nontrivially. If none of $vT, vT^2, \cdots, vT^{k-1}$ (where k is the period of T) are in E^+ then the invariant vector $v + vT + \cdots + vT^{k-1}$ is in E^- as well, but all invariant vectors are contained in F. Thus at least one $vT^j \in E^+$.

In the case $k \geq 2$ the lemma applies to $\mu_{\lambda z *}\colon T_zM \to T_zM$. Thus $\exp B'_a(z) \subset$ cl U and bdy $D \cap B_a(z) \subset$ cl D. But this contradicts Corollary 4.7. Therefore $k = 1, \lambda|$bdy $D \cap W = c$ and the proof of Lemma 4.8 is complete.

Let $x \in C_2$. There exists a neighborhood W of x in M such that $\lambda| W \cap C_1$ is continuous. Furthermore, Lemma 4.8 shows that for each point $p \in$ bpy D_i $\cap\ W$, where D_i is a component of $M - C_1$ with $\mu_{c_i}|D_i =$ id (as in Proposition 4.5), we have $\lambda p = c_i$. Define the function $h\colon W \to R$ by

$$hq = \begin{cases} c_i & \text{if } q \in \text{cl}\, D \cap W\,, \\ \lambda q & \text{if } q \in W \cap C_1\,. \end{cases}$$

The function h is clearly continuous, and $\mu_{hq}q = q$ for all $q \in W$. By Lemma 4.3, h is constant on W, which implies λ is bounded in a neighborhood of x. But this contradicts the hypothesis that C_1 is nonempty, because C_2 is dense in C_1. Thus λ is bounded on M. Proposition 4.5 then implies that μ has period c on M. Evidently, this proves the theorem.

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